these the first are the applications which come through extension of the mathematical solutions of problems in physics and chemistry to similar problems in biology. Examples of this class are the application of the mathematical laws of hydrodynamics to problems of hæmodynamics, of mathematical optics to problems of physiologic optics, and of the various mathematical expressions for stress and strain to problems of muscle and bone mechanics. This phase of the application of mathematics is of long standing in medicine, having developed in the latter part of the seventeenth century. The second class of applications arose primarily through the attempt to reduce the highly variable data of medicine to some form of graphic or numerical expression. This is a more recent development since the collection of precise medical data (other than experimental records and vital statistics) practically began with the work of Louis in the early part of the nineteenth century. The mathematical treatment of this data was first attempted by Gavarret and Quetelet a little later. The modern development of this phase has proceeded mainly along the lines laid down by Galton and Karl Pearson. Among the problems in this field at the present time are the application of the mathematical principles of variation and correlation to a large and varied mass of biologic data, and the development of suitable methods of curve fitting, coördinate analysis and establishment of empirical formulæ for this type of material.

R. M. BARTON, *Secretary-Treasurer.*

GRAPHICAL SOLUTIONS OF THE QUADRATIC, CUBIC, AND BIQUADRATIC EQUATIONS.

By T. R. RUNNING, University of Michigan.

Graphical solutions of equations may in general be divided into two classes. One class consists of geometrical constructions for each particular equation. The constructions depend upon the numerical values of the coefficients. Any such construction which will give the roots of an equation with a given set of coefficients must be changed for any equation having a different set of coefficients. The other class consists of charts or diagrams from which the roots are read when the coefficients or combinations of them are used as coördinates.

In what follows the solutions are given by means of charts. Such charts are made up of straight lines and curves from which the roots (both real and imaginary) are read approximately.

There are many such graphical solutions for the real roots of the simpler algebraic equations, but, so far as the writer is aware, the imaginary roots are obtained only by geometric constructions for each particular equation.[1] Such

[1] A few references are as follows:

Klein, *Vorträge über ausgewählte Fragen der Elementargeometrie*, 1895, pp. 28–31; in Beman and Smith's translation, p. 34.

d'Ocagne, *Traité de Nomographie*, 1899.

constructions are usually tedious to carry out when a number of equations are to be solved.

A single chart will suffice for the solution of the quadratic equation, another for the solution of the cubic, and a combination of these two charts will give the roots of the biquadratic.

Solution of the Quadratic. If in the equation

$$x^2 + Ax + B = 0 \tag{1}$$

a value is assigned to x it will represent a straight line, the coördinates of whose points are A and B. In the form

$$B = -xA - x^2 \tag{2}$$

Encyklopädie der mathematischen Wissenschaften, vol. 1, 1900–1904, pp. 1020–1050.

Brand, "Méthode graphique pour déterminer les racines réelles de l'équation $x^3 + px + q = 0$," *L'Enseignement Mathématique*, vol. 8, 1906, pp. 443–448.

Noaillon, "Résolution graphique de l'équation du troisième degré," *Mathesis*, vol. 7, 1907, pp. 122–125.

Gleason, "A simple method for graphically obtaining the complex roots of a cubic equation,". *Annals of Mathematics*, second series, vol. 11, 1910, pp. 95–96.

Lenhardt, "Graphische Darstellung reeler und komplexer Lösungen von Gleichungen," *Zeitschrift für mathematischen und naturwiss. Unterricht*, vol. 44, 1913, pp. 116–122.

Deming, "A new method for the graphical solution of algebraic equations," *Science*, vol. 43, 1916, pp. 576–580.

Hewes, "Nomograms of adjustment," *Annals of Mathematics*, second series, vol. 18, 1917, pp. 194–199.

[Some additional references, for the most part of an earlier date, are as follows:

Bérard, *Opuscules mathématiques et Méthodes nouvelles pour déterminer les racines des équations numériques*, 1811, p. 33.

Monge, "Solution graphique de l'équation du troisième degré," *Correspondance sur l'Ecole . . . Polytechnique*, vol. 3, pp. 201–203, May, 1815. (Bérard and Monge used a cubic parabola for their graphical solutions.)

Gergonne, "De la résolution des équations numériques du 3me degré par la parabole ordinaire," *Annales de Mathématiques Pures et Appliquées*, vol. 9, pp. 204–210, December, 1818.

Gerono, "Note sur la construction des racines de l'équation complète du quatrième degré," *Nouvelles Annales de Mathématiques*, vol. 3, 1844, pp. 533–536.

Scheffler, "Ueber die geometrische Construction der imaginären Wurzeln einer Gleichung," *Archiv der Mathematik und Physik*, vol. 15, 1850, pp. 375–389.

Vieille, "Sur la construction des racines de l'équation du quatrième degré par l'intersection d'une parabole et d'un cercle," *Nouvelles Annales de Mathématiques*, vol. 16, 1857, pp. 453–456.

Hoppe, "Construction der reellen Wurzeln einer Gleichung vierten oder dritten Grades mittelst einer festen Parabel," *Archiv der Mathematik und Physik*, vol. 56, 1874, pp. 110–112.

Hoppe, "Construction der imaginären Wurzeln einer Gleichung vierten oder dritten Grades mittelst einer festen Parabel," *Archiv der Mathematik und Physik*, vol. 69, 1883, pp. 216–218. (In the two papers by Hoppe the solutions are by means of a fixed parabola and a variable circle.)

Hofmann, "La solution géométrique de l'équation du quatrième degré," *Nouvelles Annales de Mathématiques*, vol. 67, 1888, pp. 120–133.

Heppel, "Quartic equations interpreted by the parabola," *Proceedings of the London Mathematical Society*, vol. 22, 1891, pp. 416–423. (Extension of, and supplement to, first of Hoppe's papers.)

Phillips and Beebe, *Graphic Algebra or Geometrical Interpretation of the Theory of Equations of one Unknown Quantity*. New York, 1904.

Irwin and Wright, "Some properties of polynomial curves," *Annals of Mathematics*, second series, vol. 19, 1917, pp. 152–158.—Editor.]

it is seen that $-x$ is the slope and $-x^2$ the intercept. Every point represented by (A, B) for real values of x will be the intersection of two lines. Such lines are drawn in Chart I and designated by the corresponding values of x.

As an illustration take the equation

$$x^2 + 2.8x - .6 = 0.$$

In order to find the roots of this equation locate the point $(2.8, -.6)$ and read the values corresponding to the lines passing through this point. The values are -3 and $.2$. In case that no line in the chart passes through the point interpolation is necessary.

Imaginary roots are determined as follows:

$$x^2 + Ax + B = 0,$$

$$x = -\tfrac{1}{2}A \pm \sqrt{(\tfrac{1}{4}A^2 - B)}.$$

Let $\frac{1}{4}A^2 - B = -K^2$, then $B = \frac{1}{4}A^2 + K^2$ represents a system of parabolas whose vertices are $(0, K^2)$, and whose axes coincide with the B-axis. These parabolas are drawn in Chart I and designated by the corresponding values of K, the coefficient of the imaginary unit i.

To find the roots of

$$x^2 + .6x + 2.69 = 0.$$

Locate the point $(.6, 2.69)$ and the real part of the roots read from the top of Chart I is $-.3$, the imaginary part read from the parabola passing through the point is $1.61i$ approximately. The roots are

$$-.3 \pm 1.61i.$$

The process of interpolation needs no explanation. If $K = 0$, that is, if

$$B = \tfrac{1}{4}A^2$$

the parabola becomes the envelope of the straight lines in the chart and forms the boundary between the region of real roots and the region of imaginary roots. This parabola is the region of equal roots.

Solution of the Cubic. The general equation of the cubic is

$$x^3 + ux^2 + vx + w = 0. \tag{3}$$

This is reduced to a cubic lacking the term of the second degree by the substitution

$$x = z + k,$$

where $k = -u/3$. Equation (3) becomes

$$z^3 + \left(v - \frac{1}{3}u^2\right)z + w - \frac{uv}{3} + \frac{2}{27}u^3 = 0. \tag{4}$$

Let $A = v - \frac{1}{3}u^2$, and $B = w - \frac{uv}{3} + \frac{2}{27}u^3$, then (4) becomes

$$B = -zA - z^3. \tag{5}$$

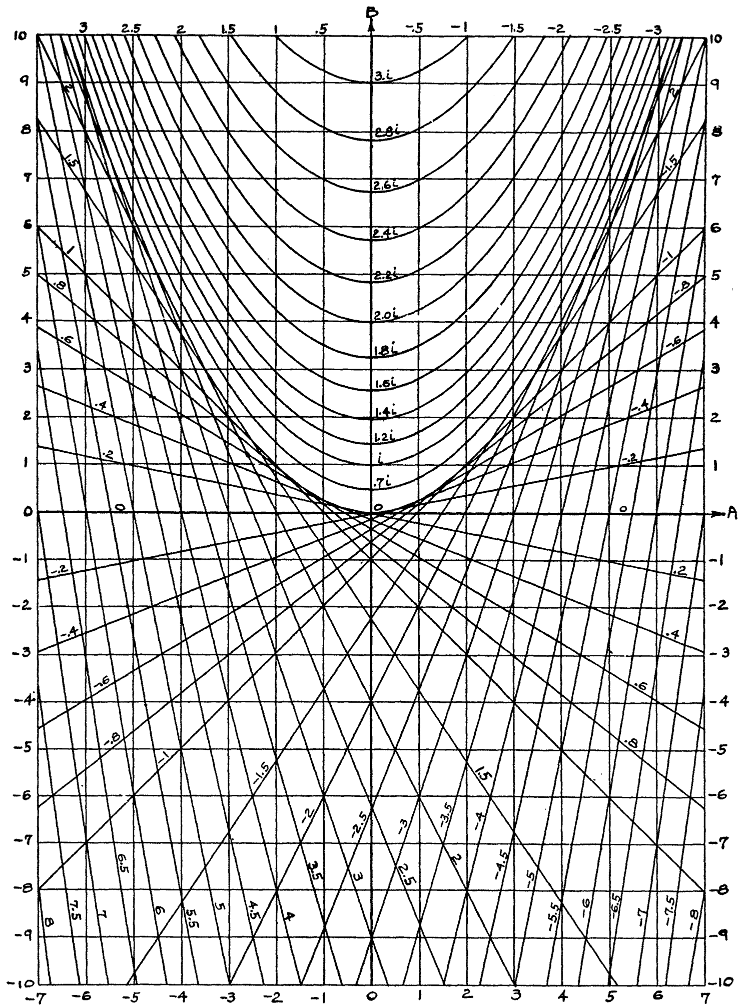

CHART I.

For given values of z equation (5) represents straight lines, the coördinates of whose points are A and B. The lines represented by this equation for different real values of z are drawn in Chart II. The real roots of a reduced cubic are the numbers designating the three lines passing through the point (A, B).[1]

[1] The method which Professor Running employs for finding the real roots of the quadratic and cubic equations was given by Lalanne in 1846. See the bibliographical footnote in connection with J. P. Ballantine's article, "A graphic solution of the cubic equation," in this MONTHLY, *1920*, 203.—EDITOR.

Take the equation

$$x^3 - 3x^2 + 1.25x + 1.5 = 0. \tag{5a}$$

By the substitution $x = z + 1$ this equation becomes

$$z^3 - 1.75z + .75 = 0. \tag{5b}$$

The roots of (5b) read from Chart II are -1.5, 1, and .5. The roots of (5a) are then $-.5$, 2, and 1.5.

In case equation (5) has imaginary roots, the three roots are of the form

$$-2a,\ a + bi, \text{ and } a - bi.$$

Also

$$A = b^2 - 3a^2, \tag{6}$$

$$B = 2a^3 + 2ab^2. \tag{7}$$

From (6)

$$a = \pm\sqrt{\frac{b^2 - A}{3}}.$$

This value of a substituted in (7) gives

$$B = \pm\frac{2}{3}\sqrt{\frac{b^2 - A}{3}}\,(4b^2 - A), \tag{8}$$

or

$$B^2 = -\frac{4}{27}(A - b^2)(A - 4b^2)^2. \tag{9}$$

For given values of b equation (9) represents cubic curves in A and B which are drawn in Chart II. For all points (A, B) along any curve the coefficient of i in the imaginary roots will be constant.

For $b = 0$ equation (9) forms the boundary between the region of imaginary roots and the region of all real roots.

As an example find the roots of

$$x^3 - 1.2x^2 + .37x - .17 = 0. \tag{10}$$

Let $x = z + .4$ and (10) becomes

$$z^3 - .11z - .15 = 0. \tag{10a}$$

The roots would be obtained by locating the point $(-.11, -.15)$ but since these coördinates are small it is better to use an equation whose roots are four times the roots of (10a) and divide by four. The equation is

$$z^3 - 1.76z - 9.6 = 0. \tag{10b}$$

The roots of this equation are 2.4, $-1.2 \pm 1.6i$, of (10a) .6, $-.3 \pm .4i$, and therefore of (10) 1, $.1 \pm .4i$. It is to be noted that the real part of the imaginary roots is one half of the negative of the real root.

An interesting property of the curves represented by (9), and which simplifies very much the process of drawing them, is the following: The segments cut off from the lines of real roots by the curves are proportional to the increments of b^2.

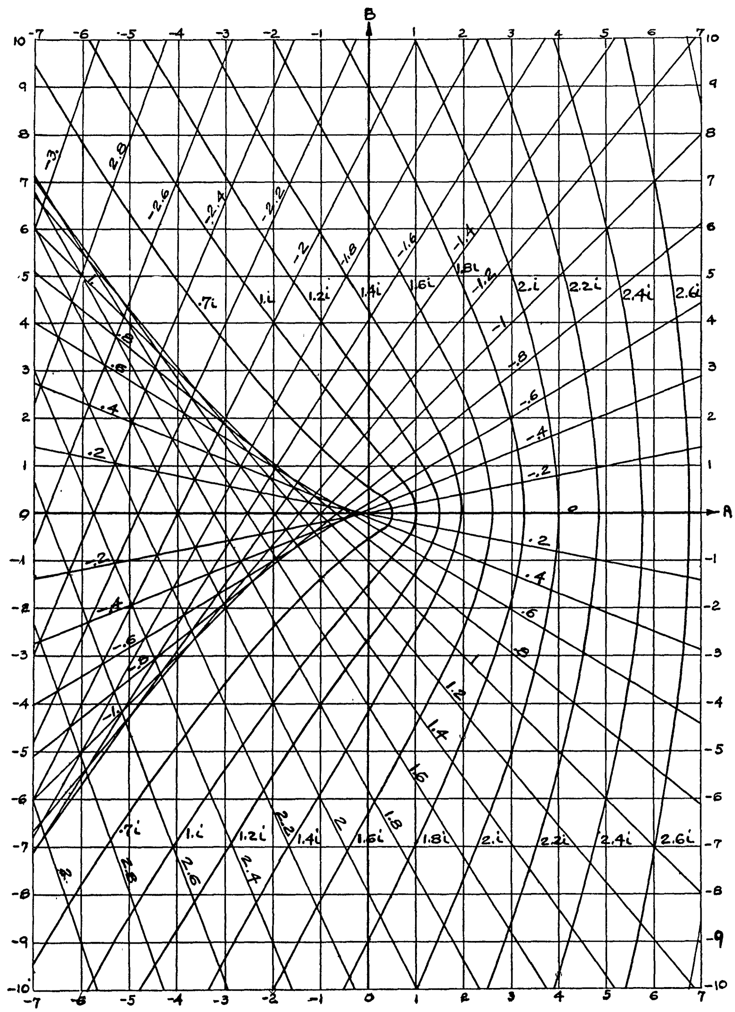

CHART II.

To show this, let A_1, B_1, and b_1 be any set of corresponding values of A, B, and b in (8). Then

$$B_1 = \pm\frac{2}{3}\sqrt{\frac{b_1^2 - A_1}{3}}\,(4b_1^2 - A_1).$$

Give to A_1 and b_1^2 the same increment K, and let B_2 be the value obtained for B.

Then

$$B_2 = \pm \frac{2}{3}\sqrt{\frac{b_1^2 - A_1}{3}}\,(4b_1^2 - A_1 + 3K),$$

and

$$B_2 - B_1 = \pm\, 2K\sqrt{\frac{b_1^2 - A_1}{3}} = 2Ka,$$

since

$$a = \pm\sqrt{\frac{b_1^2 - A_1}{3}}.$$

If now in the equation $B = - zA - z^3$ the same notation is used $B_1 = - zA_1 - z^3$, $B_2 = - zA_1 - z^3 - zK$, $B_2 - B_1 = - zK = 2Ka$, since the real root is $- 2a$. This shows that the increments of the ordinates of successive intersections of the lines representing real roots with the curves of imaginary roots are proportional to the increments of b^2 and therefore the segments are proportional to the increments. This fact enables one to locate points on the curves very rapidly. The curve $B^2 = - \frac{4}{27}A^3$ $(b = 0)$ is the envelope of the lines represented by

$$B = - zA - z^3$$

and is the discriminant of the cubic (5).

Solution of the Biquadratic. The equation of the biquadratic is

$$x^4 + Ax^3 + Bx^2 + Cx + D = 0. \tag{11}$$

It was said at the outset that the graphical solution of the biquadratic would be made to depend upon the solution of the quadratic and the cubic. The first step in the solution will be to factor the left hand member of (11) into two quadratic factors. This will be done graphically by means of Chart II together with Chart I. The biquadratic after factoring may be written

$$(x^2 + ax + b)(x^2 + cx + d) = 0$$

or

$$x^4 + (a + c)x^3 + (b + d + ac)x^2 + (ad + bc)x + bd = 0.$$

By equating coefficients the following equations are obtained

$$\left\{\begin{array}{lr} a + c = A, & b + d + ac = B, \\ ad + bc = C, & bd = D. \end{array}\right\} \tag{12}$$

Let $ac = K_1$, and $b + d = K_2$; then $K_1 + K_2 = B$. From the two equations $b + d = K_2$, and $bd = D$,

$$b = \tfrac{1}{2}K_2 + \tfrac{1}{2}\sqrt{K_2^2 - 4D},$$

$$d = \tfrac{1}{2}K_2 - \tfrac{1}{2}\sqrt{K_2^2 - 4D}.$$

From the two equations $a + c = A$, and $ac = K_1$,

$$a = \tfrac{1}{2}A + \tfrac{1}{2}\sqrt{A^2 - 4K_1},$$

$$c = \tfrac{1}{2}A - \tfrac{1}{2}\sqrt{A^2 - 4K_1}.$$

Substituting the values of a, b, c, and d above in the equation

$$ad + bc = C,$$

and remembering that $K_1 = B - K_2$, the cubic

$$K_2^3 - BK_2^2 + (AC - 4D)K_2 + 4BD - C^2 - A^2D = 0 \tag{13}$$

is obtained. By letting $K_2 = z + \frac{1}{3}B$, (13) becomes the auxiliary cubic

$$z^3 + (AC - 4D - \tfrac{1}{3}B^2)z + \tfrac{1}{3}(ABC + 8BD - \tfrac{2}{9}B^3 - 3C^2 - 3A^2D) = 0. \tag{14}$$

It is noticed that the coefficients of the auxiliary cubic (14) are obtained directly from the biquadratic (11) and its roots are found from Chart II. After obtaining the values of z, $b + d$ is found from the relation

$$K_2 = b + d = z + \tfrac{1}{3}B.$$

It is evident from the factored form of the biquadratic that there are always three possible values of $b + d$ real or imaginary. The numerically greatest of the real roots will always give real values of b and d.

Since $b + d = K_2$, and $bd = D$, it is seen that b and d are roots of

$$x^2 - K_2x + D = 0,$$

and are found from Chart I. The values of a and c are found in the same way. The values of a, b, c, and d must be selected so that they will satisfy the equation $ad + bc = C$.

As an example, consider the biquadratic equation

$$x^4 - 3x^3 + 5x^2 - x - 10 = 0.$$

The corresponding auxiliary cubic is

$$z^3 + 34.667z - 48.593 = 0.$$

Let $z = 3w$ and the equation becomes

$$w^3 + 3.85w - 1.8 = 0.$$

From Chart II: $w = .44$, $z = 1.32$, $K_2 = b + d = 2.99$, $bd = -10$. From Chart I: $b = -2$, $d = 5$ approximately; $a = -1$, and $c = -2$.

The biquadratic then becomes

$$(x^2 - x - 2)(x^2 - 2x + 5) = 0.$$

Again, from Chart I, the values of x are

$$-1, \quad 2, \quad 1 + 2i, \quad \text{and} \quad 1 - 2i.$$

The approximation to the roots when the coefficients are large or differ from one another by large numbers is illustrated by the solution of

$$x^4 - 9x^3 + 3{,}000x + 30{,}000 = 0.$$

The auxiliary cubic is

$$z^3 - 147{,}000z - 11{,}430{,}000 = 0.$$

Let $z = 200w$, then

$$w^3 - 3.675w - 1.429 = 0.$$

Chart II gives $w = 2.08$, $z = 416$. Chart I gives $b = 92$, $d = 325$, $a = 16.7$ $c = 25.5$, and $x = -8.3 \pm 5i, 12.8 \pm 13i$. The error in the value of z obtained from the chart may be several units. A good approximation for the correction is easily obtained algebraically. $b + d = 417{,}579$, $bd = 30{,}000$; therefore $b = 325.378$, $d = 92.201$.

$ac = 0 - (b + d) = -417.579$, $a + c = -9$; therefore $a = 16.424$, $c = -25.424$.

In the factored form the equation becomes

$$(x^2 + 16.424x + 92.201)(x^2 - 25.424x + 325.378) = 0.$$

Correct to three places of decimals

$$x = -8.212 \pm 4.976i, 12.712 \pm 12.798i.$$

1321, 1471, 1521, 1571, 1621, 1671, 1721, 1771, 1821.

By R. C. ARCHIBALD, Brown University.

1321—Dante Alighieri (born at Florence, 1265) died at Ravenna; his *Divine Comedy* is of mathematical importance for its statement of astronomical notions of the time.

1471—Albrecht Dürer (died, 1528), German painter and engraver, was born at Nürnberg, May 20; he was the author of various works on geometry and perspective, fortification, and human proportion ✳ In his *Zeittafeln zur Geschichte der Mathematik*, Felix Müller states[1] that Indian-Arabic numerals were in this year used for the first time in numbering leaves of a book, in Petrarch's *Liber de remediis utriusque fortunæ*, published by Arnold ther Hoernen of Cologne. This statement is erroneous, in part at least. In the library of the Annmary Brown Memorial, Providence, R. I., there is a copy of Werner Rolewinck's *Sermo in festo præsentationis* . . . ("Sermon preachable on the feast of the presentation of the blessed virgin") issued by the same publisher in 1470. It is paged in Indian-Arabic numerals half way down the right hand margins. In the printed catalogue of this library, prepared by A. W. Pollard of the British Museum, it is stated that *this* was the first printed book with pagination in Indian-Arabic numerals. The work of Petrarch mentioned above may be the *second* work of this kind. The printed catalogues of the British Museum and G. P. Winship's census of incunabula in America do not list a 1471 edition of this work of Petrarch.

[1] The same statement occurs in F. Unger, *Die Methodik der Praktischen Arithmetik in Historischer Entwickelung*, Leipzig, 1888, p. 16; and in J. Tropfke, *Geschichte der Elementar-Mathematik*, 2. Auflage, Berlin und Leipzig, 1921, p. 26.